MATHEMATICAL SCIENCES

Paper III

Time Allowed : 2½ Hours] **[Maximum Marks : 200**

Note :—This paper contains two groups, Group-I and Group-II. Attempt all questions from any one group. Each group contains four Sections. Questions in all these sections are to be answered according to the instructions given in the concerned sections.

GROUP I

SECTION I

Instructions :— This section contains *two* questions. Each question carries **20** marks.

1. Show that a subset K of $\mathbf{R}^n$ is compact iff it is closed and bounded. [20]

Or

Let V be a vector space and P, Q be linear maps of V into itself. Assume that they satisfy the following conditions :

(*a*) P + Q = I (identity map)

(*b*) PQ = QP = 0 (zero map)

(*c*) $P^2 = P$ and $Q^2 = Q$.

Show that V is the direct sum of ImP and ImQ. [20]

2. Prove that every non-zero polynomial $f(x) \in Q[x]$ can be uniquely written as a product $f(x) = cf_0(x)$, where c is a rational number and $f_0(x)$ is a primitive polynomial in $\mathbb{Z}[x]$. [20]

Or

If A and B are Banach spaces and T : A $\rightarrow$ B is an onto linear transformation, then show that the image of each open sphere centered on the origin in A contains an open sphere centered on the origin in B. [20]

SECTION II

Instructions :— This section contains *three* questions. Each question carries **15** marks.

3. Prove that energy group of order p^2(p is prime), is either cyclic or product of two cyclic groups. [15]

4. Find the equation of motion of a particle of mass m moving on the surface of a cone of semi-vertical angle θ under gravitational force. [15]

5. Let X be a non-empty set. Prove that the ring of all subsets of X is a principal ideal ring if and only if X is finite. [15]

SECTION III

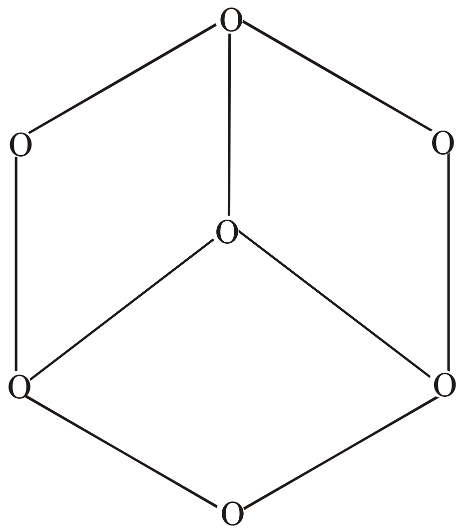
Instructions :— This section contains *nine* questions. Each question carries **10** marks.

6. Find the Fourier cosine transform of

$$f(x) = \frac{1}{1 + x^2};$$

and hence derive Fourier sine transformation of $\phi(x) = \frac{x}{1 + x^2}$. [10]

7. Is the following lattice distributive ? Justify your answer. [10]



8. Let V be a finite dimensional vector space over a field **K** and let $T \in L(V)$. If T is diagonalizable, then prove that the agebraic multiplicity of each eigen value of T equals its geometric multiplicity. [10]

9. A particle has Lagrangian

$$L = \frac{1}{2}\left[f(\theta)\dot{\theta}^2 + 2g(\theta)\,\dot{\theta}\omega + \omega^2 h(\theta)\right] - v.$$

Show that $\frac{1}{2}\left[f(\theta)\dot{\theta}^2 - \omega^2 h(\theta)\right] + v$ = constant. Explain what does this constant represent. [10]

10. Is the polynomial

$$2x^5 - 5x^4 + 5$$

solvable by radicals over Q ? Justify your answer. [10]

11. Show that the maximum or minimum of the objective function of the LPP is attained at one of the extreme points of the set of all feasible solutions. [10]

12. Prove that the preimage of a connected set under a continuous map need not be connected. [10]

13. Solve the congruence

$$x^3 + 4x + 8 \equiv 0 \pmod{15}.$$ [10]

14. A solid of revolution is formed by rotating about the x-axis, the area between the x-axis, the lines $x = 0$ and $x = 1$, and a curve through the points with the following co-ordinates.

x	y
0.00	1.000
0.25	0.9896
0.50	0.9589
0.75	0.9089
1.00	0.8415

Estimate the volume of the solid formed, giving the answer to three decimal places by using Simpson's rule. [10]

SECTION IV

Instructions :— This section contains *five* questions. Each question carries **5** marks.

15. Show that every basic feasible solution is an extreme point of the convex set of feasible solutions. [5]

16. Using the method of successive approximation solve the integral equation

$$\phi(x) = 1 + \int_0^x (x - t)\phi(t)\,dt,$$

taking $\phi_0(x) = 1$. [5]

17. Prove that every lattice with four elements is distributive. [5]

18. Show that if E is a measurable set, then each translate $E + y$ of E is also measurable. [5]

19. Let X and Y be metric spaces. Let the sequence of functions $f_n : X \to Y$ converge to f, uniformly on X. If c in a point of X at which each f_n is continuous, then show that f is continuous at c. [5]

GROUP II

SECTION I

Instructions :—This section contains *two* questions. Each question carries **20** marks.

20. Define the following modes of convergence of a sequence of random variables. State the implications between them and give a proof of any *one* of the implication.

(*i*) Convergence in probability

(*ii*) Convergence in quadratic mean

(*iii*) Convergence in distribution. [20]

Or

Let $X \sim G(\alpha_1, \beta)$ and $Y \sim G(\alpha_2, \beta)$ be independent random variables.

(*i*) Show that X + Y and X/Y are independent and obtain their distributions.

(*ii*) Show that X + Y and X/(X + Y) are independent and obtain their distributions. [20]

21. Let $X_1, X_2, ...X_n$ be a sample from $B(m, p)$. Find UMVUE of $P(1 - p)$. [20]

Or

State and prove Gauss-Markoff theorem. [20]

SECTION II

Instructions :—This section contains *three* questions. Each question carries **15** marks.

22. Consider the model $Y = X\beta + \varepsilon$ where $\varepsilon \sim N(0, \sigma^2)$

$$X = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

Verify whether $\lambda'\beta$ for λ' given below is estimable ? Justify your answer

(*i*) $\lambda' = (7, 3, 4, 2, 3, 2)$

(*ii*) $\lambda' = (6, 3, 3, 2, 2, 2)$. [15]

23. Define cluster sampling with clusters of unequal sizes. Is it possible to construct an unbiased estimator of the population mean for this sampling design ? Justify your answer with reasons. [15]

24. Let $X_1, X_2, \ldots\ldots X_n$ be iid rvs with Poisson distribution having mean λ. Let λ be distributed as $\pi(\lambda)$, where

$$\pi(\lambda) = \frac{1}{a} e^{-\lambda/a}; \ \lambda > 0, \ a > 0 .$$

Find Bayes estimate of λ under squared error loss function. [15]

SECTION III

Instructions :— This section contains *nine* questions. Each question carries **10** marks.

25. Monthly requirement schedule for a product is given by :

Month	**Requirement** (Units)
1	200
2	300
3	20
4	140
5	180
6	360
7	196
8	200
9	100
10	270

Setup cost is Rs. 360; cost of a unit is Rs. 15 and inventory carrying charges are 2.5% of the annual average inventory value. Determine an optimal plan of batch size. [10]

26. Define capability indices C_p, C_{pk} and C_{pm}. Explain situations while each of them is a useful index to decide the suitability of the process for production. [10]

27. A radioactive source emits particles according to a Poisson process of rate $\lambda = 2$ per minutes. Find the probability that :

(*i*) no particle is emitted upto 4 minutes.

(*ii*) the first particle is emitted after 4 minutes but before 6 minutes. [10]

28. Define the distance function and linkage methods used in cluster analysis. Make a comparison between hierarchical and partition-based clustering algorithms. [10]

29. Let (X, Y) be a bivariate normal random variable with parameters μ_1, μ_2, σ_1^2, σ_2^2 and ρ and let $U = aX + b$, $a \neq 0$, and $V = cY + d$, $c \neq 0$. Find the joint distribution of (U, V). [10]

30. Define Bayes risk and Bayes estimator. If X_i, $i = 1, ..., n$ is a random sample of size n from $N(\theta, 1)$ distribution and θ has $N(0, 1)$ prior, find Bayes estimator of θ and associated Bayes risk assuming the quadratic loss. [10]

31. (*i*) State the properties of the distribution function of a random variable.

    (*ii*) Let F be a distribution function of a r.v. X. Determine whether $G(x) = 1 - (1 - F(x))^2$ is a distribution function. [10]

32. If $A_1, A_2, ...A_n$ are n independent events, show that :

$$P\left(\bigcap_{j=1}^{n} A_j^c\right) \leq \exp\left(-\sum_{j=1}^{n} P(A_j)\right).$$

(Hint : Use $1 + x \leq e^x$ for all real x). [10]

33. Describe briefly the Beal's method for solving quadratic programming problem. [10]

SECTION IV

Instructions :— This section contains *five* questions. Each question carries **5** marks.

34. Define A optimality, D optimality and E optimality of a block design. [5]

35. Define systematic sampling and state the condition for systematic sampling to be more efficient than simple random sampling. [5]

36. Explain the difference between a tansportation problem and assignment problem. [5]

37. Ten units were tested at high stress test for up to 250 hours. Six failures occurred at 37, 73, 132, 195, 222 and 248 hours. Four units were taken off test without failing at the following run times : 50, 100, 200 and 250 hours. Obtain empirical cum hazard function. [5]

38. y_1, y_2, y_3 and y_4 are four independent variables with $E(y_1) = E(y_3) = \theta_1 + \theta_3 + \theta_4$ and $E(y_2) = E(y_4) = \theta_1 - \theta_2$ and $V(y_i) = \sigma^2$ for $i = 1, 2, 3, 4$. Verify whether $\theta_2 + \theta_3 + \theta_4$ is estimable. If so obtain its BLUE. [5]

ROUGH WORK

ROUGH WORK